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Three-Spin Correlation of the Ising Model on the Generalized Checkerboard Lattice

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The Ising model on the generalized checkerboard lattice is studied and the three-spin correlation function is obtained for the three nodal spins surrounding a unit cell of the checkerboard lattice. As an application of this result, the spontaneous magnetization of the internal spin within a unit cell is calculated.

KEY WORDS: Ising model; spontaneous magnetization; generalized checkerboard lattice; three-spin correlation.

1. INTRODUCTION

In 1949 Onsager announced as a conference remark the expression of the spontaneous magnetization for the square Ising lattice.[1] He never published his derivation. In 1952, Yang[2] was the first to publish a derivation, which is very complicated. The spontaneous magnetization has since been obtained for other Ising lattices, including the rectangular,[3] triangular,[4,5] honeycomb,[6] Kagomé,[6,7] checkerboard,[7,8] 4–8,[9,10] and 3–12[11,12] lattices. Recently Lin and Wu[13] considered the Ising model on the generalized checkerboard lattice which includes the 4–8 and 3–12 lattices as special cases. Using the result by Baxter and Choy[10] on a 4–8 lattice, Lin and Wu derived the spontaneous magnetization of nodal spins. Their results are expressed in terms of the Boltzmann weights of a unit cell of the checkerboard lattice without specifying its cells structures. However, they did not compute the spontaneous magnetization of internal spins within a checkerboard unit cell.

Three-spin correlation of the Ising model on a triangular lattice was first derived by Baxter[14] for three spins surrounding a triangle. A simpler

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derivation was given later by Enting.[15] Recently Baxter and Choy[16] calculated several local three-spin correlations for the square lattice free-fermion model, the equivalent checkerboard Ising model, and the triangular, honeycomb, and square lattice Ising models. Similar results were also obtained by Lin and Wu.[17] The latter authors used the result on three-spin correlations to compute the spontaneous magnetization for the Ising model on the Union Jack lattice with the most general anisotropic interactions. The purpose of this paper is to calculate the three-spin correlation of the Ising model on the generalized checkerboard lattice for three nodal spins surrounding a unit cell. Using an identity[16,17] which relates the spontaneous magnetization of the internal spin and the three-spin correlation of three nodal spins, I then calculate the spontaneous magnetization of the internal spin within a unit cell. The result is a generalization of the previous work by Lin and Chen,[18] who considered the isotropic checkerboard lattice.

The model is defined in Section 2. The three-spin correlation is derived in Section 3. In Section 4, I calculate the spontaneous magnetization of the internal spin on a generalized checkerboard lattice.

2. THE MODEL

Consider the generalized checkerboard Ising lattice shown in Fig. 1. The lattice consists of nodal spins σ_i denoted by black dots. Each shaded square is a network of internal spins connected to the rest of the lattice at the four nodal spins. Such a network is characterized by the Boltzmann weight

$$B(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = \sum_{\sigma_\alpha = \pm 1} \exp(-\beta H) \tag{1}$$

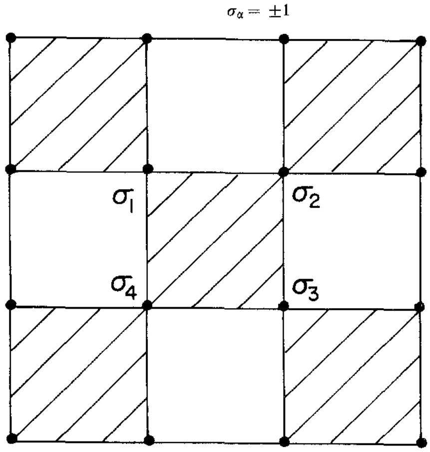

Fig. 1. The generalized checkerboard lattice.

where $\beta = 1/kT$, H is the Hamiltonian of the network, and σ_α refers to its internal spins. Assuming pairwise and noncrossing interactions, the Boltzmann weights satisfy the spin-reversal symmetry

$$B(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = B(-\sigma_1, -\sigma_2, -\sigma_3, -\sigma_4) \tag{2}$$

and the free-fermion condition[19]

$$B_1 B_2 + B_3 B_4 = B_5 B_6 + B_7 B_8 \tag{3}$$

where

$$\begin{aligned} B_1 &= B(++++), \qquad & B_2 &= B(-+-+) \\ B_3 &= B(--++), \qquad & B_4 &= B(+--+) \\ B_5 &= B(-+--), \qquad & B_6 &= B(---+) \\ B_7 &= B(+---), \qquad & B_8 &= B(--+-) \end{aligned} \tag{4}$$

It is convenient to introduce dual variables W_i, which are linear combinations of B_j:

$$2W_i = \sum_j X_{ij} B_j \tag{5}$$

where X_{ij} are elements of the matrix

$$\begin{matrix} + & + & + & + & + & + & + & + \\ + & + & + & + & - & - & - & - \\ + & + & - & - & - & - & + & + \\ + & + & - & - & + & + & - & - \\ + & - & - & + & + & - & - & + \\ + & - & - & + & - & + & + & - \\ + & - & + & - & - & + & - & + \\ + & - & + & - & + & - & + & - \end{matrix}$$

It can be shown that[20]

$$4B_i = \sum_j X_{ij} W_j \tag{6}$$

The spontaneous magnetization of the nodal spins has been calculated by Lin and Wu. The results are[13]

$$\langle \sigma_1 \rangle = \langle \sigma_3 \rangle = MF_1, \qquad \langle \sigma_2 \rangle = \langle \sigma_4 \rangle = MF_2 \tag{7}$$

where

$$M^8 = (-W_1 + W_2 + W_3 + W_4)(W_1 - W_2 + W_3 + W_4) \times (W_1 + W_2 - W_3 + W_4)(W_1 + W_2 + W_3 - W_4) \times (16 W_5 W_6 W_7 W_8)^{-1}$$

$$F_1 = [(W_5 W_7)^{1/2} + (W_6 W_8)^{1/2}]/[W_1 W_3 + W_2 W_4 + 2(W_5 W_6 W_7 W_8)^{1/2}]^{1/2}$$

$$F_2 = [(W_6 W_7)^{1/2} + (W_5 W_8)^{1/2}]/[W_1 W_4 + W_2 W_3 + 2(W_5 W_6 W_7 W_8)^{1/2}]^{1/2}$$

3. THREE-SPIN CORRELATION

The three-spin correlation $\langle \sigma_1 \sigma_2 \sigma_3 \rangle$ is invariant if we multiply the eight Bolzmann weights (4) by a common factor. The weights also satisfy the free-fermion condition (3). Therefore, only six of the eight weights are independent and we can calculate $\langle \sigma_1 \sigma_2 \sigma_3 \rangle$ if the shaded squares are realized by networks consisting of six interactions for which $\langle \sigma_1 \sigma_2 \sigma_3 \rangle$ is known.

Lin and Wu[13] pointed out that the generalized checkerboard lattice can be realized as a 4–8 lattice as shown in Fig. 2. The spontaneous magnetization of the equivalent 4–8 lattice was derived by Baxter and Choy[10] and we have

$$\langle \sigma_5 \rangle = M F_5, \qquad \langle \sigma_6 \rangle = M F_6 \tag{8}$$

where

$$F_5 = [(W_6 W_7)^{1/2} T + (W_5 W_8)^{1/2}/T] \times [W_1 W_4 + W_2 W_3 + 2(W_5 W_6 W_7 W_8)^{1/2}]^{-1/2}$$

$$F_6 = [(W_5 W_7)^{1/2} T^* + (W_6 W_8)^{1/2}/T^*] \times [W_1 W_3 + W_2 W_4 + 2(W_5 W_6 W_7 W_8)^{1/2}]^{-1/2}$$

$$T = \tanh K_2', \qquad T^* = \tanh K_1'$$

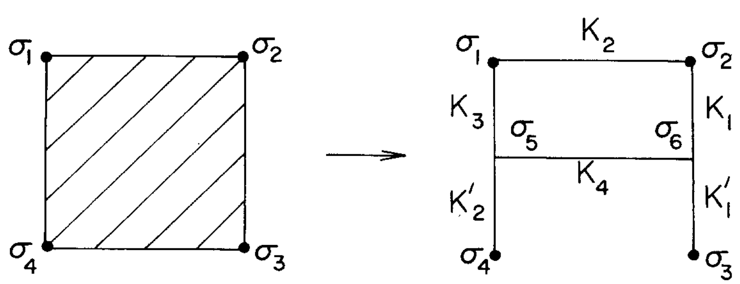

Fig. 2. Realization of the generalized checkerboard lattice as a 4–8 lattice.

The generalized checkerboard lattice can also be realized by lattices as shown in Fig. 3. We have

$$J_4 = K_2', \qquad L_3 = K_1' \tag{9}$$

because the lattices shown in Fig. 3a and 3b can be transformed by a Δ–Y transformation into the 4–8 lattice shown in Fig. 2. Consider Fig. 3a first. It can be shown that

$$\begin{aligned} T = \tanh J_4 &= \coth(J_1 + J_2 + J_3)(B_1 - B_6)/(B_1 + B_6) \\ &= \coth(-J_1 + J_2 - J_3)(B_2 - B_5)/(B_2 + B_5) \\ &= \coth(-J_1 - J_2 + J_3)(B_3 - J_8)/(B_3 + B_8) \\ &= \coth(J_1 - J_2 - J_3)(B_4 - B_7)/(B_4 + B_7) \end{aligned} \tag{10}$$

We can solve for J_i and the results are

$$\begin{aligned} \tanh 2J_1 &= 2T(B_1B_4 - B_6B_7)/[(B_1 - B_6)(B_4 - B_7) + T^2(B_1 + B_6)(B_4 + B_7)] \\ \tanh 2J_2 &= 2T(B_1B_2 - B_5B_6)/[(B_1 - B_6)(B_2 - B_5) + T^2(B_1 + B_6)(B_2 + B_5)] \\ \tanh 2J_3 &= 2T(B_1B_3 - B_6B_8)/[(B_1 - B_6)(B_3 - B_8) + T^2(B_1 + B_6)(B_3 + B_8)] \\ T^2 &= (a - b)/(a + b) \end{aligned} \tag{11}$$

where

$$\begin{aligned} a &= B_1B_5 + B_2B_6 - B_3B_7 - B_4B_8 \\ b &= 2(B_1B_2 - B_7B_8) \end{aligned}$$

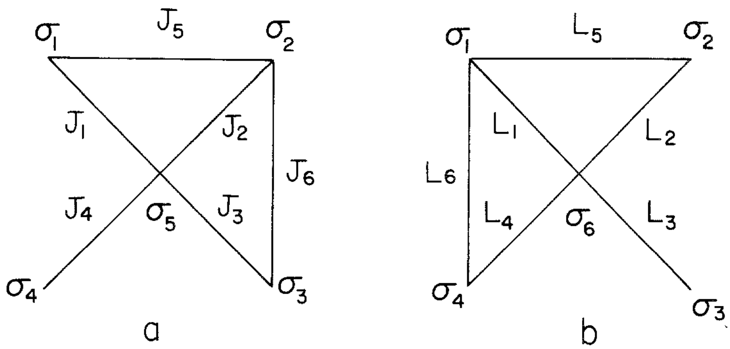

Fig. 3. Two different ways of realizing of the generalized checkerboard lattice.

Similar results can be obtained for L_i by reflecting Fig. 3a about the vertical line, exchanging B_5 with B_7, and B_6 with B_8.

To calculate the three-spin correlation, we use the following identity,[16,17] which is a generalization of that used by Choy and Baxter[21] to anisotropic interactions:

$$\begin{aligned}\langle\sigma_5\rangle &= \langle\tanh(J_1\sigma_1+J_2\sigma_2+J_3\sigma_3+J_4\sigma_4)\rangle\\ &=\lambda_1\langle\sigma_1\rangle+\lambda_2\langle\sigma_2\rangle+\lambda_3\langle\sigma_3\rangle+\lambda_4\langle\sigma_4\rangle+\mu_1\langle\sigma_2\sigma_3\sigma_4\rangle+\mu_2\langle\sigma_3\sigma_4\sigma_1\rangle\\ &\quad+\mu_3\langle\sigma_4\sigma_1\sigma_2\rangle+\mu_4\langle\sigma_1\sigma_2\sigma_3\rangle \end{aligned} \tag{12}$$

where

$$\lambda_1=(A+B)/8, \qquad \mu_1=(A-B)/8$$

$$\begin{aligned} A &= \tanh(J_1+J_2+J_3+J_4)+\tanh(J_1+J_2-J_3-J_4)\\ &\quad+\tanh(J_1-J_2-J_3+J_4)+\tanh(J_1-J_2+J_3-J_4)\\ B &= \tanh(J_1-J_2-J_3-J_4)+\tanh(J_1-J_2+J_3+J_4)\\ &\quad+\tanh(J_1+J_2-J_3+J_4)+\tanh(J_1+J_2+J_3-J_4)\end{aligned}$$

and other λ_i and μ_i can be obtained from λ_1 and μ_1 by cyclically permuting 1, 2, 3, 4. We define

$$S_i=\langle\sigma_j\sigma_k\sigma_l\rangle/M, \qquad i\neq j\neq k\neq l \tag{13}$$

and rewrite (12) in the form

$$\mu_1 S_1+\mu_2 S_2+\mu_3 S_3+\mu_4 S_4=F_5-(\lambda_1+\lambda_3)F_1-(\lambda_2+\lambda_4)F_2 \tag{14}$$

where

$$\begin{aligned}\mu_1 &= (B_2/B_5+B_5/B_2+B_3/B_8+B_8/B_3-B_1/B_6-B_6/B_1-B_4/B_7-B_7/B_4)\\ &\quad\times(8\sinh 2J_4)^{-1}\\ \mu_2 &= (B_3/B_8+B_8/B_3+B_4/B_7+B_7/B_4-B_1/B_6-B_6/B_1-B_2/B_5-B_5/B_2)\\ &\quad\times(8\sinh 2J_4)^{-1}\\ \mu_3 &= (B_2/B_5+B_5/B_2+B_4/B_7+B_7/B_4-B_1/B_6-B_6/B_1-B_3/B_8-B_8/B_3)\\ &\quad\times(8\sinh 2J_4)^{-1}\\ \mu_4 &= (B_1/B_6-B_6/B_1+B_2/B_5-B_5/B_2+B_3/B_8-B_8/B_3+B_4/B_7-B_7/B_4)\\ &\quad\times(8\sinh 2J_4)^{-1}\end{aligned}$$

$$\lambda_1+\lambda_3=(B_1/B_6-B_6/B_1-B_2/B_5+B_5/B_2)/4\sinh 2J_4$$

$$\lambda_2+\lambda_4=\coth 2J_4-(B_3/B_8+B_4/B_7+B_5/B_2+B_6/B_1)/4\sinh 2J_4$$

We define

$$G_1 = [(W_6 W_8)^{1/2} - (W_5 W_7)^{1/2}]/[W_1 W_3 + W_2 W_4 + 2(W_5 W_6 W_7 W_8)^{1/2}]^{1/2}$$
$$G_2 = [(W_5 W_8)^{1/2} - (W_6 W_7)^{1/2}]/[W_1 W_4 + W_2 W_3 + 2(W_5 W_6 W_7 W_8)^{1/2}]^{1/2} \tag{15}$$

Substituting the expression (8) for F_5 into (14), we get

$$a_1(S_1 + S_3) + b_1(S_2 + S_4) + c_1(S_1 - S_3) + d_1(S_2 - S_4) = 2e_1 F_1 + 2f_1 F_2 + 8G_2 \tag{16}$$

where

$$a_1 = B_2/B_5 + B_5/B_2 - B_1/B_6 - B_6/B_1$$
$$b_1 = B_3/B_8 + B_4/B_7 - B_5/B_2 - B_6/B_1$$
$$c_1 = B_3/B_8 + B_8/B_3 - B_4/B_7 - B_7/B_4$$
$$d_1 = B_7/B_4 + B_8/B_3 - B_1/B_6 - B_2/B_5$$
$$e_1 = B_2/B_5 - B_5/B_2 + B_6/B_1 - B_1/B_6$$
$$f_1 = B_3/B_8 + B_4/B_7 + B_5/B_2 + B_6/B_1$$

Similarly, we have the identity

$$\langle \sigma_6 \rangle = \langle \tanh(L_1 \sigma_1 + L_2 \sigma_2 + L_3 \sigma_3 + L_4 \sigma_4) \rangle \tag{17}$$

Following exactly the same procedure, we get

$$a_2(S_1 + S_3) + b_2(S_2 + S_4) + c_2(S_1 - S_3) + d_2(S_2 - S_4) = 2e_2 F_1 + 2f_2 F_2 + 8G_1 \tag{18}$$

Notice that (18) can be obtained from (16) by exchanging B_5, B_6, W_3, W_5, S_1, S_3, and F_1, respectively, with B_7, B_8, W_4, W_6, S_2, S_4, and F_2.

Reflecting the lattices shown in Fig. 3 about the horizontal line, we get two more independent equations from (16) and (18) by exchanging B_5, B_6, W_3, W_7, S_1, S_2, and F_1, respectively, with B_8, B_7, W_4, W_8, S_4, S_3, and F_2:

$$a_3(S_1 + S_2) + b_3(S_2 + S_4) + c_3(S_1 - S_3) + d_3(S_2 - S_4) = 2e_3 F_1 + 2f_3 F_2 - 8G_1 \tag{19}$$

$$a_4(S_1 + S_2) + b_4(S_2 + S_4) + c_4(S_1 - S_3) + d_4(S_2 - S_4) = 2e_4 F_1 + 2f_4 F_2 - 8G_2 \tag{20}$$

Our goal is to calculate S_i from four linear equations (16) and (18)–(20). After a lengthy calculation, we finally get

$$\begin{aligned} S_1 &= F_1 + 2(p_1 F_1 + q_1 G_1 + r_1 F_2 + s_1 G_2)/E \\ S_2 &= F_2 + 2(p_2 F_2 + q_2 G_2 + r_2 F_1 + s_2 G_1)/E \\ S_3 &= F_1 + 2(p_3 F_1 - q_3 G_1 + r_3 F_2 - s_3 G_2)/E \\ S_4 &= F_2 + 2(p_4 F_2 - q_4 G_2 + r_4 F_1 - s_4 G_1)/E \end{aligned} \tag{21}$$

where

$$\begin{aligned} E &= W_5 W_6 W_7 W_8 - W_1 W_2 W_3 W_4 \\ p_1 &= (B_1 B_2 - B_7 B_8)(B_5^2 + B_6^2) - B_5 B_6(B_1^2 + B_2^2 + B_7^2 + B_8^2) \\ &\quad + (B_1 B_7 + B_2 B_8)(B_4 B_5 + B_3 B_6) \\ q_1 &= (B_1 B_2 - B_7 B_8)(B_3 B_5 + B_4 B_6 - B_1 B_7 - B_2 B_8) \\ &\quad + (B_1 B_8 + B_2 B_7)(B_3 B_4 + B_5 B_6) \\ &\quad - (B_1 B_2 + B_7 B_8)(B_3 B_6 + B_4 B_5) \\ r_1 &= B_3 B_4(B_7^2 - B_8^2) + B_5 B_6(B_1^2 - B_2^2) + (B_2 B_8 - B_1 B_7)(B_3 B_6 + B_4 B_5) \\ s_1 &= (B_1 B_2 - B_7 B_8)[(B_3 - B_4)(B_7 + B_8) + (B_6 - B_5)(B_1 + B_2)] \end{aligned}$$

S_2 is obtained from S_1 by the exchange of B_5 with B_7, and B_6 with B_8. One obtains S_3 by the exchange of B_5 with B_6, and B_7 with B_8. One obtains S_4 by the exchange of B_5 with B_8, and B_6 with B_7.

In the special case of the checkerboard lattice with four interactions (see Fig. 4), we have

$$S_1 = 1 - [\exp(-2P^*) + \exp(-2J_2 - 2J_3)]/\sinh 2J_2 \sinh 2J_3 \tag{22}$$

where

$$\begin{aligned} \cosh 2P^* &= (W_6 W_8 + W_5 W_7)/(W_6 W_8 - W_5 W_7) \\ &= (\cosh 2J_1 \cosh 2J_4 \sinh 2J_2 \sinh 2J_3 \\ &\quad + \sinh 2J_1 \sinh 2J_4 \cosh 2J_2 \cosh 2J_3) \\ &\quad \times (\sinh 2J_2 \sinh 2J_3 - \sinh 2J_1 \sinh 2J_4)^{-1} \end{aligned}$$

Equation (22) was first obtained by Baxter and Choy.[16] Their derivation is based on the concept of the Z-invariant,[22] which cannot be applied to an arbitrary generalized checkerboard lattice.

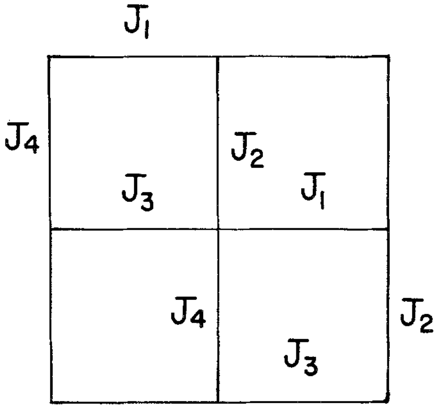

Fig. 4. A checkerboard lattice with four interactions.

In the special case of an isotropic generalized checkerboard lattice we have

$$B_3 = B_4, \qquad B_5 = B_6 = B_7 = B_8$$

$$W_3 = W_4, \qquad W_5 = W_6 = W_7 = W_8$$

$$\begin{aligned} S/F_1 &= 1 - 4B_5^2/(B_1 - B_3)^2 \\ &= 1 - (W_1 - W_2)^2/4(W_3 + W_5)^2 \end{aligned} \tag{23}$$

which was first derived by Lin and Chen.[18]

In the special case of $B_5 = B_6$, $B_7 = B_8$, we have

$$W_5 = W_6, \qquad W_7 = W_8$$

$$S_1 = F_1 + 4TB_5(W_5 W_7)^{1/2}/R_1 R_2[W_5 W_7 + (W_1 W_2 W_3 W_4)^{1/2}] \tag{24}$$

where

$$R_1 = (W_1 W_3 + W_2 W_4 + 2W_5 W_7)^{1/2}$$

$$R_2 = (W_1 W_4 + W_2 W_3 + 2W_5 W_7)^{1/2}$$

$$\begin{aligned} T = {} & (W_1 W_4 - W_2 W_3)(W_5 + W_7)/[R_1 + (W_1 W_3)^{1/2} + (W_2 W_4)^{1/2}] \\ & + (W_1 W_4 - W_2 W_3)[(W_1 W_3)^{1/2} + (W_2 W_4)^{1/2}] \\ & \times [W_5 + W_7 + (W_1 W_2)^{1/2} + (W_3 W_4)^{1/2}]^{-1} \\ & + [W_2 W_3(W_1 + W_4) - W_1 W_4(W_2 + W_3) - 4W_5 W_7 B_5] \\ & \times [R_2 + (W_1 W_4)^{1/2} + (W_2 W_3)^{1/2}]^{-1} \\ & - 2B_5[(W_1 W_4)^{1/2} + (W_2 W_3)^{1/2}] \end{aligned}$$

4. SPONTANEOUS MAGNETIZATION

Consider a generalized checkerboard lattice where the Hamiltonian of the unit cell includes multispin interactions which involve only an even number of spins. The spontaneous magnetization $\langle\sigma\rangle$ of the internal spin σ is a function of the Boltzmann weights

$$B(\sigma_1, \sigma_2, \sigma_3, \sigma_4 \mid \sigma) = B(-\sigma_1, -\sigma_2, -\sigma_3, -\sigma_4 \mid -\sigma) = \sum_{\sigma_\alpha = \pm 1}' \exp(-\beta H) \tag{25}$$

where the prime means summing over all internal spins except σ. We define

$$\begin{aligned}
&B_1^\pm = B(++++\mid\pm), && B_2^\pm = B(+-+-\mid\pm)\\
&B_3^\pm = B(++--\mid\pm), && B_4^\pm = B(+--+\mid\pm)\\
&B_5^\pm = B(+-++\mid\pm), && B_6^\pm = B(+++-\mid\pm)\\
&B_7^\pm = B(+---\mid\pm), && B_8^\pm = B(++-+\mid\pm)\\
&R_i = (B_i^+ - B_i^-)/(B_i^+ + B_i^-)
\end{aligned} \tag{26}$$

The network characterized by the 16 Boltzmann weights (25) is equivalent to a star network as shown in Fig. 5 with ten pairwise and five four-spin interactions such that

$$B(\sigma_1, \sigma_2, \sigma_3, \sigma_4 \mid \sigma) = \rho \exp(E) \tag{27}$$

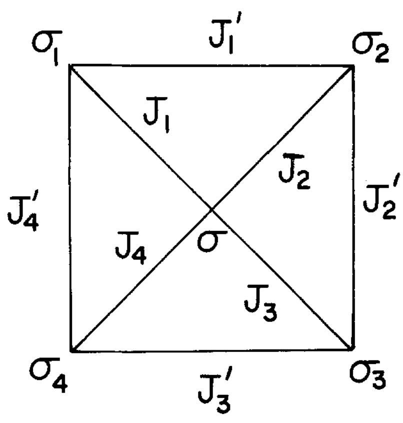

Fig. 5. A star network with two-spin and four-spin interactions (only pairwise nearest-neigbor interactions are shown).

where

$$E = \sigma(J_1\sigma_1 + J_2\sigma_2 + J_3\sigma_3 + J_4\sigma_4 + K_1\sigma_2\sigma_3\sigma_4 + K_2\sigma_3\sigma_4\sigma_1 + K_3\sigma_4\sigma_1\sigma_2 + K_4\sigma_1\sigma_2\sigma_3) + J'_1\sigma_1\sigma_2 + J'_2\sigma_2\sigma_3 + J'_3\sigma_3\sigma_4 + J'_4\sigma_4\sigma_1 + L\sigma_1\sigma_3 + L'\sigma_2\sigma_4 + K\sigma_1\sigma_2\sigma_3\sigma_4$$

It follows from (27) that we have the identity

$$\begin{aligned}\langle\sigma\rangle &= \langle\tanh(J_1\sigma_1 + J_2\sigma_2 + J_3\sigma_3 + J_4\sigma_4 + K_1\sigma_2\sigma_3\sigma_4 + K_2\sigma_3\sigma_4\sigma_1 + K_3\sigma_4\sigma_1\sigma_2 + K_4\sigma_1\sigma_2\sigma_3)\rangle \\ &= \sum_{i=1}^{4} \lambda_i\langle\sigma_i\rangle + \mu_1\langle\sigma_2\sigma_3\sigma_4\rangle + \mu_2\langle\sigma_3\sigma_4\sigma_1\rangle + \mu_3\langle\sigma_4\sigma_1\sigma_2\rangle + \mu_4\langle\sigma_1\sigma_2\sigma_3\rangle \end{aligned} \tag{28}$$

where

$$\lambda_1 = (A+B)/8, \qquad \mu_1 = (A-B)/8$$

$$\begin{aligned} A = {} & \tanh(J_1 + J_2 + J_3 + J_4 + K_1 + K_2 + K_3 + K_4) \\ & + \tanh(J_1 - J_2 + J_3 - J_4 + K_1 - K_2 + K_3 - K_4) \\ & + \tanh(J_1 + J_2 - J_3 - J_4 + K_1 + K_2 - K_3 - K_4) \\ & + \tanh(J_1 - J_2 - J_3 + J_4 + K_1 - K_2 - K_3 + K_4) \end{aligned}$$

$$\begin{aligned} B = {} & \tanh(J_1 - J_2 - J_3 - J_4 - K_1 + K_2 + K_3 + K_4) \\ & + \tanh(J_1 - J_2 + J_3 + J_4 - K_1 + K_2 - K_3 - K_4) \\ & + \tanh(J_1 + J_2 - J_3 + J_4 - K_1 - K_2 + K_3 - K_4) \\ & + \tanh(J_1 + J_2 + J_3 - J_4 - K_1 - K_2 - K_3 + K_4) \end{aligned}$$

and other λ_i and μ_i are obtained from λ_1 and μ_1 by cyclically permuting 1, 2, 3, 4. When $K_i = 0$, (28) reduces to (12). After some algebra we obtain

$$\lambda_i = (a_i + b_i)/8, \qquad \mu_i = (a_i - b_i)/8 \tag{29}$$

where

$$\begin{aligned} a_1 &= R_1 + R_2 + R_3 + R_4, & b_1 &= R_5 + R_6 + R_7 + R_8 \\ a_2 &= R_1 - R_2 + R_3 - R_4, & b_2 &= -R_5 + R_6 - R_7 + R_8 \\ a_3 &= R_1 + R_2 - R_3 - R_4, & b_3 &= R_5 + R_6 - R_7 - R_8 \\ a_4 &= R_1 - R_2 - R_3 + R_4, & b_4 &= R_5 - R_6 - R_7 + R_8 \end{aligned}$$

When the Boltzmann weights satisfy the condition (3), the one-spin and three-spin correlations of the nodal spins are given by (7) and (21) and we have

$$\langle \sigma \rangle = M\left[(\lambda_1 + \lambda_3) F_1 + (\lambda_2 + \lambda_4) F_2 + \sum_{i=1}^{4} \mu_i S_i \right] \qquad (30)$$

Fisher[23] proved that a triangular network of interactions with spin-reversal symmyetry is equivalent to a triangle with three pairwise interactions. Therefore a checkerboard unit cell consisting of one or several triangular networks always satisfies the condition (3). The special case of the triangular checkerbroard lattice was studied by Lin.[24]

5. SUMMARY

I have obtained the three-spin correlation of the Ising model for the three nodal spins surrounding a unit cell of the generalized checkerboard lattice. The result is expressed in terms of Boltzmann weights of a unit cell of the checkerboard lattice without specifying its cell structure. The central theme of the calculation is the use of (12), in which $\langle \sigma_5 \rangle$ is known from ref. 16 and the other one-spin correlations are known from ref. 17. The four unknown three-spin correlations are then obtained by solving (12) and three similar equations obtained by appropriate permutations of indices. The result is given by (21).

I have considered the Ising model on a generalized checkerboard lattice and derived the spontaneous magnetization of the internal spin within a unit cell. The spontaneous magnetization is a linear combination of the three-spin correlations. The result is given by (30).

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